

New Properties of Controlled Continuous K - g -Frames in Hilbert C^* -Modules

M'hamed Ghiati¹, Mohamed Rossafi², Samir Kabbaj¹

¹Departement of Mathematics, Ibn Tofail University, B. P. 133, Kenitra, Morocco

²LASMA Laboratory Department of Mathematics Faculty of Sciences, Dhar El Mahraz University Sidi Mohamed Ben Abdellah, Fes, Morocco

Correspondence should be addressed to M'hamed Ghiati: mhamed.ghiati@uit.ac.ma

Email(s): mhamed.ghiati@uit.ac.ma (M. Ghiati), mohamed.rossafi@usmba.ac.ma (M. Rossafi), samkabbaj@yahoo.fr (S. Kabbaj)

Abstract

Frame theory in Hilbert spaces were introduced by Duffin and Schaeffer in 1952 to study some deep problems in non-harmonic Fourier series. This theory has a great revolution for recent years. It has been extended from Hilbert Space to Hilbert C^* -modules. In this paper, we study some New Properties of (C, C') -controlled continuous K - g -Frames in Hilbert C^* -modules and we established some equalities for controlled continuous K - g -Frames.

Keywords: controlled continuous- K - g -frame; continuous K - g -frame; Hilbert C^* -modules.

Introduction and Preliminaries

Frame theory in Hilbert spaces is a new theory which was introduced by Duffin and Schaeffer [8] in 1952 to study some deep problems in nonharmonic Fourier series. This theory was reintroduced and developed by Daubechies, Grossman and Meyer [6].

In 1993, S. T. Ali, J. P. Antoine and J. P. Gazeau [1] introduced the concept of continuous frames in Hilbert spaces. Gabardo and Han in [11] called these kinds of frames, frames associated with measurable spaces.

In 2000, Frank and Larson [10] introduced the notion of frame in Hilbert C^* -modules as a generalization of frame in Hilbert spaces. The theory of continuous frames has been generalized in Hilbert C^* -modules. For more details, see [2, 23, 28].

The aim of this paper is to extend results, given for Hilbert C^* -module in discret case.

Let U and V be tow Hilbert C^* -modules and $\{V_m\}_{m \in M}$ is a sequence of subspaces of V , where M is a subset of Z and $End_{\mathcal{A}}^*(U, V_m)$ is the collection of all adjointable \mathcal{A} -linear maps from U into V_m , $\mathcal{GL}(\mathcal{U})$ is the set of all bounded linear operators which have bounded inverses and $\mathcal{GL}^+(\mathcal{U})$ is the set of all positive operators in $\mathcal{GL}(\mathcal{U})$, and $\mathcal{C}, \mathcal{C}' \in \mathcal{GL}^+(\mathcal{U})$.

Let us recall basic definitions and notations of controlled continuous- K - g -frames in Hilbert C^* -modules.

Definition 1.1. [21] Let \mathcal{A} be a unital C^* -algebra and U be a left \mathcal{A} -module, such that the linear structures of \mathcal{A} and \mathcal{U} are compatible. U is a pre-Hilbert \mathcal{A} -module if U is equipped with an \mathcal{A} -valued in product $\langle \cdot, \cdot \rangle_{\mathcal{A}}: \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,

- (i) $\langle \alpha f + \beta g, h \rangle_{\mathcal{A}} = \alpha \langle f, h \rangle_{\mathcal{A}} + \beta \langle g, h \rangle_{\mathcal{A}}$ for all $f, g, h \in \mathcal{U}$ and $\alpha, \beta \in C$,
- (ii) $\langle af, g \rangle_{\mathcal{A}} = a \langle f, g \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $f, g \in \mathcal{U}$,
- (iii) $\langle f, g \rangle_{\mathcal{A}} = \langle g, f \rangle_{\mathcal{A}}^*$ for all $f, h \in \mathcal{U}$,
- (iv) $\langle f, f \rangle_{\mathcal{A}} \geq 0$ for all $f \in \mathcal{U}$ and $\langle f, f \rangle_{\mathcal{A}} = 0$ if and only if $f = 0$.

For $f \in \mathcal{U}$ we define a norm on \mathcal{U} , $\|f\| = \|\langle f, f \rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If \mathcal{U} is complete with $\|\cdot\|$, it is called a Hilbert \mathcal{A} -module or a Hilbert C^* -module over \mathcal{A} .

For every a in C^* -algebra \mathcal{A} we have $|a| = (a^*a)^{\frac{1}{2}}$ and the \mathcal{A} -valued norm on \mathcal{H} is defined by $|f| = \langle f, f \rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{U}$.

Let \mathcal{U} and \mathcal{K} be two Hilbert \mathcal{A} modules,

A map $T : \mathcal{U} \rightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \rightarrow \mathcal{U}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $y \in \mathcal{K}$ and $x \in \mathcal{U}$.

Lemma 1.2. [2] Let A be a C^* -algebra, U and V two Hilbert \mathcal{A} module, and $T \in \text{End}_{\mathcal{A}}^*(U, V)$. Then the following statements are equivalent:

- (i) T is surjective,
- (ii) T^* is bounded below with respect to norm, that is, $m \geq 0$, such that $\|T^*\| \geq m\|f\|$ for all $f \in U$,
- (iii) T^* is bounded below with respect to the inner product, that is, $m' \geq 0$ such that $\langle T^*f, T^*f \rangle \geq m'\|f\|$, for all $f \in U$.

Lemma 1.3. [29] Let (Ω, μ) be a measure space, X and Y are two Banach spaces, $\lambda : X \rightarrow Y$ be a bounded linear operator and $f : \Omega \rightarrow Y$ measurable function, then

$$\lambda\left(\int_{\Omega} f d\mu\right) = \int_{\Omega} (\lambda f) d\mu.$$

Lemma 1.4. [7] Suppose that V_1 and V_2 two Hilbert \mathcal{A} -Modules \mathcal{U} and $L_1 \in \text{End}_{\mathcal{A}}^*(V_1, U)$, $L_2 \in \text{End}_{\mathcal{A}}^*(V_2, U)$. Then the following assertions are equivalent:

- (i) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$
- (ii) $L_1 L_1^* \leq \lambda^2 L_2 L_2^*$ for some $\lambda > 0$;
- (iii) There exists a mapping $T \in \text{End}_{\mathcal{A}}^*(V_1, V_2)$ such that $L_1 = L_2 T$.

Moreover, if above conditions are valid, then there exists a unique operator U such that

- (i) $\|U\|^2 = \inf\{\alpha > 0 : L_1 L_1^* \leq \alpha L_2 L_2^*\},$
- (ii) $\ker(L_1) = \ker(U),$
- (iii) $\mathcal{R}(U) \subseteq \overline{\mathcal{R}(L_2^*)}.$

If an operator T has a closed range, then there exists a right-inverse operator T^\dagger , (pseudo-inverse of T) in the following sense.

Lemma 1.5. [28] Let $T \in \text{End}_{\mathcal{A}}^*(V_1, V_2)$ be a bounded operator with closed range $\mathcal{R}(T)$. Then there exists a bounded operator $T^\dagger \in \text{End}_{\mathcal{A}}^*(V_2, V_1)$ for which

$$TT^\dagger x = x, \quad x \in \mathcal{R}(T).$$

Lemma 1.6. [21] Let \mathcal{U} and \mathcal{V} two Hilbert \mathcal{A} -module and $T \in \text{End}_{\mathcal{A}}^*(U, V)$. Then, the following assertions are equivalent:

- (i) The operator T is bounded and \mathcal{A} -linear.

(ii) There exist $k > 0$ such that $\langle Tx, Tx \rangle_{\mathcal{A}} \leq k \langle x, x \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$.

Lemma 1.7. [27] Let U be a Hilbert C^* -module. If $P, Q \in \text{End}_{\mathcal{A}}^*(U, U)$ are two bounded A -linear operators in H and $P + Q = I_H$, the one has

$$P - P^*P = Q^* - Q^*Q.$$

Lemma 1.8. [27] Let U be a Hilbert C^* -module. If $P, Q \in \text{End}_{\mathcal{A}}^*(U, U)$ are two bounded, selfadjoint A -linear operators in H and $P + Q = I_U$, the one has

$$\langle Pf, f \rangle + |Qf|^2 = \langle Qf, f \rangle + |Pf|^2 \geq \frac{3}{4} \langle f, f \rangle$$

.

Lemma 1.9. [27] Let U be a Hilbert C^* -module. If T is a bounded, selfadjoint linear operator and satisfy $\langle Tf, f \rangle = 0$, for all $f \in U$, then $T = 0$.

Definition 1.10. A family $\Lambda := \{\Lambda_m \in \text{End}_{\mathcal{A}}^*(U, V_m)\}_{m \in M}$ is called a Continuous g -frame in Hilbert \mathcal{A} module \mathcal{U} with respect to $\{V_m\}_{m \in M}$ if there exist constants $0 < A \leq B < +\infty$ such that for each $f \in \mathcal{U}$,

$$A \langle f, f \rangle_{\mathcal{A}} \leq \int_M \langle \Lambda_m f, \Lambda_m f \rangle_{\mathcal{A}} d\mu(m) \leq B \langle f, f \rangle_{\mathcal{A}}.$$

For this frames the g frame operator is defined by

$$S_{\Lambda}(f) = \int_M \Lambda_m^* \Lambda_m f d\mu(m), \quad f \in \mathcal{U}$$

which is positive and invertible.

Definition 1.11. Let $C \in \mathcal{GL}(\mathcal{U})$. We say that $F := \{f_m\}_{m \in M}$ is a C -controlled frame in Hilbert \mathcal{A} -module \mathcal{U} if there exist constants $0 < A_c < B_c < +\infty$ such that for each $f \in \mathcal{U}$,

$$A_c \langle f, f \rangle_{\mathcal{A}} \leq \int_M \langle f, f_m \rangle \langle C f_m, f \rangle_{\mathcal{A}} d\mu(m) \leq B_c \langle f, f \rangle_{\mathcal{A}} \quad (1.1)$$

Definition 1.12. Let $C, C' \in \mathcal{GL}^+(\mathcal{U})$, we say that $\Lambda := \{\Lambda_m \in \text{End}_{\mathcal{A}}^*(U, V_m)\}_{m \in M}$ is a (C, C') -controlled continuous K - g -frame in Hilbert \mathcal{A} -module U if there exist constants $0 < A_{cc'} < B_{cc'} < +\infty$ such that for each $f \in U$,

$$A_{cc'} \langle K^* f, K^* f \rangle_{\mathcal{A}} \leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \leq B_{cc'} \langle f, f \rangle_{\mathcal{A}} \quad (1.2)$$

If the right hand of holds, Λ is called a (C, C') -controlled- K - g -Bessel sequence in Hilbert \mathcal{A} -module \mathcal{U} with bound B_c .

We call Λ a Parseval (C, C') -controlled Continuous K - g -frame if

$$\langle K^* f, K^* f \rangle_{\mathcal{A}} = \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m).$$

If $K = I_U$. Then Λ is (C, C') -controlled continuous g -frame.

Let $\Lambda := \{\Lambda_m \in \text{End}_{\mathcal{A}}^*(U, V_m)\}_{m \in M}$ is a $\mathcal{C}, \mathcal{C}'$ -controlled continuous- g -frame in Hilbert \mathcal{A} -module U with respect to $\{V_m\}_{m \in M}$.

Define the controlled continuous- g -frame operator $S_{CC'}$ on U by:

$$S_{CC'}(f) := \int_M C^* \Lambda_m^* \Lambda_m C' f d\mu(m)$$

Proposition 1.13. *The frames operator $S_{CC'}$ is a positive, bounded and invertible.*

Proof. For any $f \in U$ we have

$$\begin{aligned}\langle S_{CC'}f, f \rangle &= \left\langle \int_M C^* \Lambda_m^* \Lambda_m C' f d\mu(m), f \right\rangle, \\ &= \int_M \langle C^* \Lambda_m^* \Lambda_m C' f, f \rangle d\mu(m), \\ &= \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle d\mu(m).\end{aligned}$$

So

$$A\langle f, f \rangle_{\mathcal{A}} \leq \langle S_{CC'}f, f \rangle \leq B\langle f, f \rangle_{\mathcal{A}}.$$

Or in the notation from operator theory $AI \leq S_{CC'} \leq BI$, thus $S_{CC'}$ is a positive operator. Furthermore, $0 \leq A^{-1}S_{\Lambda} - I \leq (\frac{B-A}{A})I$, and consequently

$$\|A^{-1}S_{\Lambda} - I\| \leq \|(\frac{B-A}{A})I\| \leq 1.$$

This shows that $S_{CC'}$ is invertible. Now we show that $S_{CC'}$ is a bounded operator

$$\|S_{CC'}\| = \sup_{\|f\| \leq 1} \langle S_{CC'}f, f \rangle = \sup_{\|f\| \leq 1} \left\| \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle d\mu(m) \right\| \leq B$$

□

Some characterizations of controlled continuous K - g -frames in Hilbert C^* -modules

In this section, we will characterize the equivalencies of controlled continuous K - g -frames in Hilbert C^* -modules from several aspects. Now, we can define the synthesis and analysis operators of the CC' -controlled continuous g -frames.

Theorem 1.14. *Let $\{\Lambda_m \in \text{End}_{\mathcal{A}}^*(U, V_m)\}_{m \in M}$ for any $\{m \in M\}$. Then $\{\Lambda_m : m \in M\}$ is a $\mathcal{C}, \mathcal{C}'$ -controlled continuous K - g -frame in Hilbert \mathcal{A} -module U with respect to $\{V_m\}_{m \in M}$ if and only if there exist constants $A, B \geq 0$ such that for any $f \in \mathcal{U}$,*

$$A\|K^*f\|^2 \leq \left\| \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \right\| \leq B\|f\|^2. \quad (1.3)$$

Proof. Let $\{\Lambda_m : m \in M\}$ is a $\mathcal{C}, \mathcal{C}'$ -controlled continuous K - g -frame in Hilbert \mathcal{A} -module U with respect to $\{V_m\}_{m \in M}$. Then

$$\begin{aligned}A\langle K^*f, K^*f \rangle_{\mathcal{A}} &\leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \leq B\langle f, f \rangle_{\mathcal{A}}, \\ A\|K^*f, K^*f\|_{\mathcal{A}} &\leq \left\| \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \right\| \leq B\|f, f\|_{\mathcal{A}}\end{aligned}$$

If inequality (1.3) holds, then by Proposition 1.13

$$\langle S^{1/2}f, S^{1/2}f \rangle = \langle Sf, f \rangle = \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m).$$

Hence

$$\begin{aligned}\sqrt{A}\|K^*f\| &\leq \|S^{1/2}f\| \leq \sqrt{B}\|f\| \\ A_1\langle K^*f, K^*f \rangle_{\mathcal{A}} &\leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \leq B_1\langle f, f \rangle_{\mathcal{A}}.\end{aligned}$$

Which implies that $\{\Lambda_m : m \in M\}$ is a $\mathcal{C}, \mathcal{C}'$ -controlled continuous K -g-frame in Hilbert \mathcal{A} -module U with respect to $\{V_m\}_{m \in M}$.

Now, let

$$\mathcal{R} := \{(C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f : f \in \mathcal{U}\}_{m \in M} \subset \left(\sum_{m \in M} \oplus H \right)_{\ell^2}.$$

It is easy to check that \mathcal{R} is a closed of $(\sum_{m \in M} \oplus H)_{\ell^2}$.

For any $f = \{f_m : m \in M\}$ and $g = \{g_m : m \in M\}$, if the Λ -valued inner product is defined by $\langle f, g \rangle = \int_M \langle f_m, g_m \rangle d\mu(m)$, the norm is defined by $\|f\|^2 = \|\langle f, f \rangle\|$. Now, we can define the synthesis and analysis operators of the CC' -controlled g -frames as

$$T_{\mathcal{C}\mathcal{C}'} : \mathcal{R} \rightarrow \mathcal{U},$$

$$T_{\mathcal{C}\mathcal{C}'}((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)_{m \in M} = \int_m (C^* \Lambda_m^* \Lambda_m C' f) d\mu(m),$$

and

$$T_{\mathcal{C}\mathcal{C}'}^* : \mathcal{U} \rightarrow \mathcal{R},$$

$$T_{\mathcal{C}\mathcal{C}'}^*(f) = ((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)_{m \in M}.$$

Thus, the CC' -controlled g -frame operator is given by

$$\begin{aligned} S_{\mathcal{C}\mathcal{C}'}(f) &= T_{\mathcal{C}\mathcal{C}'} T_{\mathcal{C}\mathcal{C}'}^*(f) \\ &= \int_M (C^* \Lambda_m^* \Lambda_m C' f) d\mu(m). \end{aligned}$$

□

Lemma 1.15. Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$. A sequence Λ is a CC' -controlled continuous g -Bessel sequence in Hilbert \mathcal{A} -module with bound $B_{\mathcal{C}\mathcal{C}'}$ if and only if the operator

$$T_{\mathcal{C}\mathcal{C}'} : \mathcal{R} \rightarrow \mathcal{H},$$

$$T_{\mathcal{C}\mathcal{C}'}((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)_{m \in M} = \int_M (C^* \Lambda_m^* \Lambda_m C' f) d\mu(m)$$

is well-defined and bounded with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$.

Proof. We only need to prove the sufficient condition. Let $T_{\mathcal{C}\mathcal{C}'}$ be a well-defined and bounded operator with $\|T_{CC'}\| \leq \sqrt{B_{CC'}}$. For each $f \in H$, we have

$$\begin{aligned} \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) &= \int_M \langle C^* \Lambda_m^* \Lambda_m C' f, f \rangle_{\mathcal{A}} d\mu(m) \\ &= \left\langle \int_M C^* \Lambda_m^* \Lambda_m C' f, f \right\rangle_{\mathcal{A}} d\mu(m) \\ &= \langle T_{CC'}((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)_{m \in M}, f \rangle_{\mathcal{A}}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\langle T_{CC'}((C^* \Lambda_i^* \Lambda_i C')^{\frac{1}{2}} f)_{i \in I}, f \rangle_{\mathcal{A}}\| &\leq \|T_{CC'}((C^* \Lambda_i^* \Lambda_i C')^{\frac{1}{2}} f)_{i \in I}\| \|f\| \\ &\leq \|T_{CC'}\| \|((C^* \Lambda_i^* \Lambda_i C')^{\frac{1}{2}} f)_{i \in I}\| \|f\|. \end{aligned}$$

But

$$\|((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)\|^2 = \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle d\mu(m)$$

$$\begin{aligned}\|((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)\| &\leq \|T_{CC'}\| \|f\|, \\ \|((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f)\|^2 &\leq \|T_{CC'}\|^2 \|f\|^2.\end{aligned}$$

It follows that

$$\int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \leq B_{CC'} \|\langle f, f \rangle_{\mathcal{A}}\|.$$

and this means that Λ is a CC' -controlled continuous g -Bessel sequence. \square

Lemma 1.16. *Let $C, C' \in \mathcal{GL}^+(\mathcal{H})$. A sequence Λ is a CC' -controlled continuous g -frame sequence in Hilbert \mathcal{A} -module if and only if the operator*

$$\begin{aligned}T_{\mathcal{CC}'} : \mathcal{R} &\rightarrow \mathcal{H}, \\ T_{\mathcal{CC}'}((C^* \Lambda_m^* \Lambda_m C')^{\frac{1}{2}} f) &= \int_M (C^* \Lambda_m^* \Lambda_m C' f) d\mu(m)\end{aligned}$$

is well-defined, bounded and surjective.

Proof. Suppose that Λ is a CC' -controlled continuous g -frame in Hilbert \mathcal{A} -module. Since, $S_{\mathcal{CC}'}$ is surjective operator, so $T_{\mathcal{CC}'}$. For the opposite implication, by Lemma 1.15; $T_{\mathcal{CC}'}$ is a well-defined and bounded operator. So Λ is a CC' -controlled continuous g -Bessel sequence. Now, for each $f \in H$, we have $f = T_{CC'} T_{CC'}^\dagger f$. Hence

$$\begin{aligned}\|f\|^4 &= \|\langle f, f \rangle\|^2 \\ &= \|\langle T_{CC'} T_{CC'}^\dagger f, f \rangle\|^2 \\ &= \|\langle T_{CC'}^\dagger f, T_{CC'}^* f \rangle\|^2 \\ &\leq \|\langle T_{CC'}^\dagger f, T_{CC'}^\dagger f \rangle\|^2 \|\langle T_{CC'}^* f, T_{CC'}^* f \rangle\|^2 \\ &\leq \|T_{CC'}^\dagger f\|^2 \|T_{CC'}^* f\|^2 \\ &\leq \|T_{CC'}^\dagger\|^2 \|f\|^2 \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m).\end{aligned}$$

We conclude that

$$(\|T_{CC'}^\dagger\|^2)^{-1} \|\langle f, f \rangle\| \leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m).$$

\square

Operators preserving controlled K - g -frames

In this section, for the CC' -controlled continuous K - g -frames $\{\Lambda_m\}_{m \in M}$. We consider some proper relations between the operators $W, K \in B(U)$ and $C, C' \in \mathcal{GL}^+(\mathcal{U})$ and investigate the cases that $\{\Lambda_m W\}_{m \in M}$, $\{\Lambda_m W^*\}_{m \in M}$ can also CC' -controlled continuous K - g -frame. Next, by putting connections between the operators $\{S_\Lambda\}, K, C$ and C' we reach to necessary and sufficient conditions that $\{\Lambda_m\}_{m \in M}$ can be a Parseval CC' -controlled continuous K - g -frames.

Theorem 1.17. *Let Λ be a CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U} and $W \in B(U)$ be a co-isometry (i.e. $WW^* = Id_U$) such that $WK = KW$ and W^* commutes with C and C' . Then $(\Lambda_m W^*)_{m \in M}$ is a CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U} .*

Proof. Suppose that Λ is a CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U} with frame bounds $A_{CC'}$. And $B_{CC'}$ for each $f \in \mathcal{U}$, we have

$$\begin{aligned}\int_M \langle \Lambda_m W^* C' f, \Lambda_m W^* C f \rangle_{\mathcal{A}} d\mu(m) &= \int_M \langle \Lambda_m C' W^* f, \Lambda_m C W^* f \rangle_{\mathcal{A}} d\mu(m) \\ &\leq B_{CC'} \langle W^* f, W^* f \rangle_{\mathcal{A}}\end{aligned}$$

hence,

$$\int_M \langle \Lambda_m W^* C' f, \Lambda_m W^* C f \rangle_{\mathcal{A}} d\mu(m) \leq B_{CC'} \|W^*\|^2 \langle f, f \rangle_{\mathcal{A}}$$

So, $(\Lambda_m^* W^*)_{m \in M}$ is a CC' -controlled continuous g -Bessel sequence. For the lower bound, we can write

$$\begin{aligned} \int_M \langle \Lambda_m W^* C' f, \Lambda_m W^* C f \rangle_{\mathcal{A}} d\mu(m) &= \int_M \langle \Lambda_m C' W^* f, \Lambda_m C W^* f \rangle_{\mathcal{A}} d\mu(m) \\ &\geq A_{CC'} \langle K^* W^* f, K^* W^* f \rangle_{\mathcal{A}} \\ &= A_{CC'} \langle (WK)^* f, (WK)^* f \rangle_{\mathcal{A}} \\ &= A_{CC'} \langle (KW)^* f, (KW)^* f \rangle_{\mathcal{A}} \\ &= A_{CC'} \langle W^* K^* f, W^* K^* f \rangle_{\mathcal{A}} \\ &= A_{CC'} \langle K^* f, K^* f \rangle_{\mathcal{A}} \end{aligned}$$

□

Theorem 1.18. Let $\Lambda := \{\Lambda_m \in B(U, V_m)_{m \in M}\}$ and $\Theta := \{\Theta_m \in B(U, V_m)_{m \in M}\}$ be two CC' -controlled continuous K - g -Bessel sequences in Hilbert \mathcal{A} -module \mathcal{U} with bounds B_Λ and B_Θ respectively. suppose that $T_{\Lambda, C, C'}$ and $T_{\Theta, CC'}$ are their synthesis operators such that $T_{\Theta, C, C'} T_{\Lambda, C, C'}^* = K^*$. Then Λ and Θ are CC' -controlled continuous K and K^* - g -frames, respectively.

Proof.

$$\begin{aligned} \|K^* f\|^4 &= \|\langle K^* f, K^* f \rangle\|^2 \\ &= \|\langle T_{\Theta, C, C'} T_{\Lambda, C, C'}^* f, K^* f \rangle\|^2 \\ &\leq \|T_{\Lambda, C, C'}^* f\|^2 \|T_{\Theta, C, C'} K^* f\|^2 \\ &= \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) \int_M \langle \Theta_m C' K^* f, \Theta_m C' K^* f \rangle_{\mathcal{A}} d\mu(m) \\ &\leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) B_\Theta \|\langle K^* f, K^* f \rangle_{\mathcal{A}}\| \end{aligned}$$

So,

$$\|\langle K^* f, K^* f \rangle\| \leq B_\Theta \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m)$$

Thus

$$B_\Theta^{-1} \|\langle K^* f, K^* f \rangle_{\mathcal{A}}\| \leq \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m)$$

This that Λ is a CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U} with frame operator S_Λ . For each $f \in \mathcal{U}$, we have $T_{\Lambda, C, C'} T_{\Theta, C, C'}^* = K$. Thus

$$B_\Lambda^{-1} \|\langle K f, K f \rangle_{\mathcal{A}}\| \leq \int_M \langle \Theta_m C' f, \Theta_m C f \rangle_{\mathcal{A}} d\mu(m)$$

This that Θ is a CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U} . □

Theorem 1.19. Let Λ be a g -frame in Hilbert \mathcal{A} -module \mathcal{U} with frame operator S_Λ . Also assume that Λ is a CC' -controlled continuous g -Bessel sequence with frame operator $S_{CC'}$. Then Λ is a Parseval CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U} if and only if $C = (S_\Lambda^{-p})^* \Phi$ and $C' = (S_\Lambda^{-q}) \Psi$ where Φ, Ψ are two operators in Hilbert \mathcal{A} -module \mathcal{U} such that $\Phi^* \Psi = KK^*$ and $p + q = 1$ where $p, q \in \mathbb{R}$.

Proof. Assume that Λ is a Parseval CC' -controlled continuous K - g -frame in Hilbert \mathcal{A} -module \mathcal{U}

$$\begin{aligned} \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathcal{A}} d\mu(m) &= \langle K^* f, K^* f \rangle_{\mathcal{A}} \\ &= \int_M \langle f, C^* \Lambda_m^* \Lambda_m C' f \rangle_{\mathcal{A}} d\mu(m) \\ &= \langle f, \int_M C^* \Lambda_m^* \Lambda_m C' f \rangle_{\mathcal{A}} d\mu(m) \\ &= \langle f, S_{CC'} f \rangle_{\mathcal{A}} \\ &= \langle f, KK^* f \rangle_{\mathcal{A}} \end{aligned}$$

$$\begin{aligned} S_{CC'}(f) &= \int_M C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \\ &= C^* \int_M \Lambda_m^* \Lambda_m C' (f) d\mu(m) \\ &= C^* S_{\Lambda} C' (f). \end{aligned}$$

Hence $S_{CC'} = C^* S_{\Lambda} C'$ and $S_{CC'} = KK^*$. Therefore, for each $p, q \in \mathbb{R}$ such that $p + q = 1$, we obtain

$$KK^* = C^* S_{\Lambda}^p S_{\Lambda}^q C'.$$

We define $\Phi = (S_{\Lambda}^p)^* C$ and $\Psi = (S_{\Lambda}^q)^* C'$ So

$$\Phi^* \Psi = C^* S_{\Lambda}^p S_{\Lambda}^q C' = KK^*.$$

Conversely, let Φ and Ψ be tow operators in Hilbert \mathcal{A} - module \mathcal{U} such that $\Phi^* \Psi = KK^*$. Suppose that $C = (S_{\Lambda}^p)^* \Phi$ and $C' = (S_{\Lambda}^q)^* \Psi$ are tow operators on Hilbert \mathcal{A} -module \mathcal{H} where $p, q \in \mathbb{R}$ and $p + q = 1$, Since

$$KK^* = \Phi^* \Psi = C^* S_{\Lambda}^p S_{\Lambda}^q C' = C^* S_{\Lambda} C' = S_{CC'}.$$

So, for each $f \in \mathcal{H}$,

$$\langle KK^* f, f \rangle_{\mathcal{A}} = \langle K^* f, K^* f \rangle_{\mathcal{A}} = \langle \int_M C^* \Lambda_m^* \Lambda_m C' f, f \rangle_{\mathcal{A}} d\mu(m).$$

Thus Λ is Parseval CC' -controlled continuous k - g -frame on Hilbert \mathcal{A} - module \mathcal{U} . □

Some Equalities for CC' -controlled Continuous K - g -frames in Hilbert C^* -Modules

Some equalities for frames involving the real parts of some complex numbers have been established in [12]. These inequalities generalized in [27] for g -frame in Hilbert C^* -modules. In this section, we generalize the equalities to a more general form which generalized before equalities and we deduce some equalities for Controlled Continuous K - g -frames in Hilbert C^* - modules.

Proposition 1.20. *Let $\{\Lambda_m : m \in M\}$ be a CC' controlled continuous g -frame for U with respect to $\{V_m : m \in M\}$ with controlled continuous g -frame operator S_{Λ} with bounds A and B . Then $\{\Lambda_m : m \in M\}$ defined by $\widetilde{\Lambda}_m = \Lambda_m S^{-1}$ is a controlled continuous g -frame for U with respect to $\{V_m : m \in M\}$ with controlled continuous g -frame operator S_{Λ}^{-1} with bounds B^{-1} and A^{-1} . That is called controlled continuous canonical dual g -frame of $\{\Lambda_m : m \in M\}$*

Proof. Let \widetilde{S} be the controlled continuous g -frame operator associated with $\{\Lambda_m : m \in M\}$ that is $\widetilde{S}f = \int_M C^* \widetilde{\Lambda}_m^* \widetilde{\Lambda}_m C' f d\mu(m)$. Then for $f \in U$,

$$\begin{aligned} S\widetilde{S}f &= \int_M C^* \widetilde{\Lambda}_m^* \widetilde{\Lambda}_m C' f d\mu(m) \\ &= \int_M C^* S S^{-1} \Lambda_m^* \Lambda_m C' S^{-1} f d\mu(m) \\ &= \int_M C^* \Lambda_m^* \Lambda_m C' S^{-1} f d\mu(m) \\ &= S S^{-1} f = f. \end{aligned}$$

Hence $\widetilde{S} = S^{-1}$.

Since $\{\Lambda_m : m \in M\}$ is a controlled continuous g -frame for U , then $AI \leq S \leq BI$. On other hand since I and S are selfadjoint and S^{-1} commutative with I and S , $AIS^{-1} \leq SS^{-1} \leq BIS^{-1}$, and hence $B^{-1}I \leq S^{-1} \leq A^{-1}I$. \square

Remark 1.21. We have $\widetilde{\Lambda}_m S^{-1} = \Lambda_m S^{-1} S = \Lambda_m$. In other words $\{\Lambda_m : m \in M\}$ and $\{\widetilde{\Lambda}_m : m \in M\}$ are dual controlled continuous g -frame with respect to each other.

Theorem 1.22. Let $\{\Lambda_m \in \text{End}_{\mathfrak{A}}^*(U, V_m) : m \in M\}$ be a CC' controlled continuous K - g -frame, for Hilbert C^* -module U with respect to $\{V_m : m \in M\}$ and let $\{\widetilde{\Lambda}_m : m \in M\}$ be the CC' canonical dual controlled continuous k - g frame of $\{\Lambda_m : m \in M\}$, then for any measurable subset $J \subset M$ and $f \in U$, one has

$$\begin{aligned} \int_J \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) + \int_M \langle \widetilde{\Lambda}_m S_{J^c} C' f, \widetilde{\Lambda}_m S_{J^c} C' f \rangle_{\mathfrak{A}} d\mu(m) &= \\ \int_{J^c} \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) + \int_M \langle \widetilde{\Lambda}_m S_J C' f, \widetilde{\Lambda}_m S_J C' f \rangle_{\mathfrak{A}} d\mu(m) &\geq \\ \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) &\geq \frac{3}{4} \langle K^* f, K^* f \rangle_{\mathfrak{A}} \end{aligned}$$

Proof. Since S is an invertible, positive operator on U , and $S_J + S_{J^c} = S$, then $S^{-1/2} S_J S^{-1/2} + S^{-1/2} S_{J^c} S^{-1/2} = I_{\mathfrak{U}}$. Let $P = S^{-1/2} S_J S^{-1/2}$, $Q = S^{-1/2} S_{J^c} S^{-1/2}$. By Lemma 1.8, we obtain

$$\langle S^{-1/2} S_J S^{-1/2} f, f \rangle + |S^{-1/2} S_{J^c} S^{-1/2} f|^2 = \langle S^{-1/2} S_{J^c} S^{-1/2} f, f \rangle + |S^{-1/2} S_{J^c} S^{-1/2} f|^2 \geq \frac{3}{4} \langle f, f \rangle.$$

Replacing f by $S^{1/2} f$, then one has

$$\langle S_J f, f \rangle + \langle S^{-1} S_{J^c} f, S^{J^c} f \rangle = \langle S_{J^c} f, f \rangle + \langle S^{-1} S_J f, S^J f \rangle \geq \frac{3}{4} \langle S f, f \rangle.$$

On the other hand, we have

$$\begin{aligned} \langle S_J f, f \rangle &= \left\langle \int_J C^* \Lambda_m^* \Lambda_m C' f d\mu(m), f \right\rangle_{\mathfrak{A}} = \int_J \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m), \\ \int_J \langle \widetilde{\Lambda}_m C' f, \widetilde{\Lambda}_m C' f \rangle_{\mathfrak{A}} d\mu(m) &= \int_J \langle \Lambda_m S^{-1} C' f, \Lambda_m S^{-1} C f \rangle_{\mathfrak{A}} d\mu(m), \\ &= \int_J \langle C^* \Lambda_m^* \Lambda_m S^{-1} C' f, S^{-1} f \rangle_{\mathfrak{A}} d\mu(m) = \langle S S^{-1} f, S^{-1} f \rangle_{\mathfrak{A}} = \langle f, S^{-1} f \rangle_{\mathfrak{A}} = \langle S^{-1} f, f \rangle_{\mathfrak{A}}. \end{aligned}$$

\square

Theorem 1.23. Let $\{\Lambda_m \in \text{End}_{\mathfrak{A}}^*(U, V_m) : m \in M\}$ be a CC' controlled continuous parseval K - g -frame, for Hilbert C^* -module U with respect to $\{V_m : m \in M\}$ then for any measurable subset $J \subset M$ and $f \in U$, one has

$$\int_J \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) + \left| \int_{J^c} C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \right|^2$$

$$\begin{aligned}
&= \int_{J^c} \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) + \left| \int_J C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \right|^2 \\
&\geq \frac{3}{4} \langle K^* f, K^* f \rangle_{\mathfrak{A}}
\end{aligned}$$

Proof. Let $\{\Lambda_m \in \text{End}_{\mathfrak{A}}^*(U, V_m) : m \in M\}$ is a CC' controlled continuous Parseval K -g-frame, for Hilbert C^* -module U with respect to $\{V_m : m \in M\}$ then for any $f \in U$, we have

$$\int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) = \langle K^* f, K^* f \rangle_{\mathfrak{A}} \quad (1.4)$$

So

$$\langle Sf, f \rangle = \left\langle \int_M C^* \Lambda_m^* \Lambda_m C' f d\mu(m), f \right\rangle = \int_M \langle \Lambda_m C' f, \Lambda_m C f \rangle_{\mathfrak{A}} d\mu(m) = \langle K^* f, K^* f \rangle_{\mathfrak{A}}$$

Hence for any $f \in \mathcal{U}$, we have $\langle (S - I_U), f \rangle_{\mathfrak{A}} = 0$. Let $T = S - I_U$. Since S is bounded, selfadjoint, then $T^* = (S - I_U)^* = S^* - I_U^* = S - I_U = T$, So T is also bounded, selfadjoint. By Lemma 1.9, we have $T = 0$, namely, $S = I_U$, so $\widetilde{\Lambda_m} = \Lambda_m S^{-1} = \Lambda_m$. From (1.4), then we have that for any measurable subset $J \subset M$ and $f \in \mathcal{U}$,

$$\begin{aligned}
\int_M \langle \widetilde{\Lambda_m} S_J C' f, \widetilde{\Lambda_m} S_J C' f \rangle_{\mathfrak{A}} d\mu(m) &= \int_M \langle \Lambda_m S_J C' f, \Lambda_m S_J C' f \rangle_{\mathfrak{A}} d\mu(m) \\
&= \langle S_J f, S_J f \rangle_{\mathfrak{A}} = \left| \int_J C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \right|^2.
\end{aligned}$$

$$\begin{aligned}
\int_M \langle \widetilde{\Lambda_m} S_{J^c} C' f, \widetilde{\Lambda_m} S_{J^c} C' f \rangle_{\mathfrak{A}} d\mu(m) &= \int_M \langle \Lambda_m S_{J^c} C' f, \Lambda_m S_{J^c} C' f \rangle_{\mathfrak{A}} d\mu(m) \\
&= \langle S_{J^c} f, S_{J^c} f \rangle_{\mathfrak{A}} = \left| \int_{J^c} C^* \Lambda_m^* \Lambda_m C' f d\mu(m) \right|^2,
\end{aligned}$$

Combining (1.4) and Theorem 1.22, we get the result. \square

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